Three-Dimensional Polyhedra can be Described by Three Polynomial Inequalities*

Gennadiy Averkov and Martin Henk

Abstract

Bosse et al. conjectured that for every natural number $d \geq 2$ and every d-dimensional polytope P in $\mathbb{R}^d$ there exist d polynomials $p_0(x), \ldots, p_{d-1}(x)$ satisfying $P = \{x \in \mathbb{R}^d : p_0(x) \geq 0, \ldots, p_{d-1}(x) \geq 0\}$. We show that for dimensions $d \leq 3$ even every d-dimensional polyhedron can be described by d polynomial inequalities. The proof of our result is constructive.

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1 Introduction

A subset S of $\mathbb{R}^d$ ($d \in \mathbb{N}$) is said to be an *elementary closed semi-algebraic set* in $\mathbb{R}^d$ if

$$S = (q_1, \ldots, q_m)_{\geq 0} := \{x \in \mathbb{R}^d : q_1(x) \geq 0, \ldots, q_m(x) \geq 0\} \tag{1.1}$$

for some $m \in \mathbb{N}$ and polynomials $q_1(x), \ldots, q_m(x)$ over $\mathbb{R}$. Every elementary closed semi-algebraic set S in $\mathbb{R}^d$ can be represented by $d\,(d+1)/2$ polynomial inequalities, i.e., there exist polynomials $p_1(x), \ldots, p_n(x)$ with $n \leq d\,(d+1)/2$ such that

$$S = (p_1, \ldots, p_n)_{\geq 0}. \tag{1.2}$$

This follows from the well-known result of Bröcker and Scheiderer; see [BCR98, Sections 6.5 and 10.4], [ABR96, p. 143]. However, the arguments of Bröcker and Scheiderer are highly non-constructive, so that not much is known about the relationship of the polynomials $q_1(x), \ldots, q_m(x)$ and $p_1(x), \ldots, p_n(x)$, not to mention possible algorithms of determination of $p_1(x), \ldots, p_n(x)$ from $q_1(x), \ldots, q_m(x)$. One can consider the following more specific problem. Given a special class $\mathcal{S}$ of elementary closed semi-algebraic sets in $\mathbb{R}^d$, find the minimal n such that every semi-algebraic set S from $\mathcal{S}$ can be represented by n polynomials $p_1(x), \ldots, p_n(x)$ and, moreover, find algorithms for determination of $p_1(x), \ldots, p_n(x)$ from $q_1(x), \ldots, q_m(x)$; see also [Hen07], [Ave08].

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Results on representations of semi-algebraic sets are relevant for several research areas including polynomial and semi-definite optimization; see [Lau08], [HN08]. In this manuscript we are concerned with polynomial representations of polyhedra. A non-empty subset P of $\mathbb{R}^d$ is said to be a *polyhedron* if P is the intersection of a finite number of closed halfspaces; see [Zie95], [Gru07]. Furthermore, P is called a *polytope* if it is a convex hull of a finite point set in $\mathbb{R}^d$. It is well-known that polytopes can be characterized as bounded polyhedra. The main result of the paper is the following theorem.

Theorem 1.1. *Let $d \in \{2, 3\}$. Then there exists an algorithm that takes a d-dimensional polyhedron P in $\mathbb{R}^d$ and constructs d polynomials $p_0(x), \dots, p_{d-1}(x) \in \mathbb{R}[x]$ satisfying $P = (p_0, \dots, p_{d-1})_{\geq 0}$.* □

Bosse, Grötschel, and Henk [BGH05] conjectured that, for every $d \geq 2$, d polynomial inequalities suffice for representing a d-dimensional polytope. Thus, Theorem 1.1 confirms the above mentioned conjecture for $d \leq 3$. We remark that the conjecture has recently been confirmed for simple polytopes of any dimension; see [AH07], [Ave08]. Recall that a d-dimensional polytope P is said to be *simple* if every vertex of P is contained in precisely d facets.

The paper is organized as follows. Section 2 contains a sketch of the construction of the polynomials $p_0(x), \dots, p_{d-1}(x)$ from Theorem 1.1 and indicates some proof ideas. Sections 3 and 4 present preliminaries from real algebraic geometry and convex geometry, respectively. In Section 5 we introduce and study a special family of neighborhoods of a given convex polytope, which is used in the main proof. Section 6 is concerned with approximation of polytopes and polyhedral cones by algebraic surfaces. In Section 7 we prove Theorem 1.1.

2 A sketch of the construction

We give a sketch of the construction of the polynomials $p_0(x), \dots, p_{d-1}(x)$ from Theorem 1.1. Standard notions from convexity that are used in this section are introduced in Section 3 or can be found in the monographs [Sch93], [Gru07], [Zie95].

Let $d \leq 3$ and let P be a d-dimensional polyhedron in $\mathbb{R}^d$.

2.1 The case of polygons

Consider the case when P is a convex polygon in $\mathbb{R}^2$ (i.e., $d = 2$ and P is bounded). Let m be the number of edges of P. Then $P = \{x \in \mathbb{R}^d : q_1(x) \geq 0, \dots, q_m(x) \geq 0\}$ for appropriate affine functions $q_1(x), \dots, q_m(x)$. We define $p_2(x) := q_1(x) \cdot \ldots \cdot q_m(x)$. One can construct a strictly concave polynomial $p_0(x)$ vanishing on each vertex of P. A possible construction of $p_0(x)$ was first given by Bernig; see [Ber98]. The set $(p_0)_{\geq 0}$ is a strictly convex body whose boundary contains all vertices of P. It is intuitively clear that for $p_0(x)$ and $p_1(x)$ as above the equality $P = (p_0, p_1)_{\geq 0}$ is fulfilled; see also Fig. 1.

2.2 The case of 3-polytopes

Consider the case when P is a 3-polytope in $\mathbb{R}^3$ (i.e., $d = 3$ and P is bounded). Let m be the number of facets of P. Then $P = \{x \in \mathbb{R}^d : q_1(x) \geq 0, \dots, q_m(x) \geq 0\}$ for appropriate

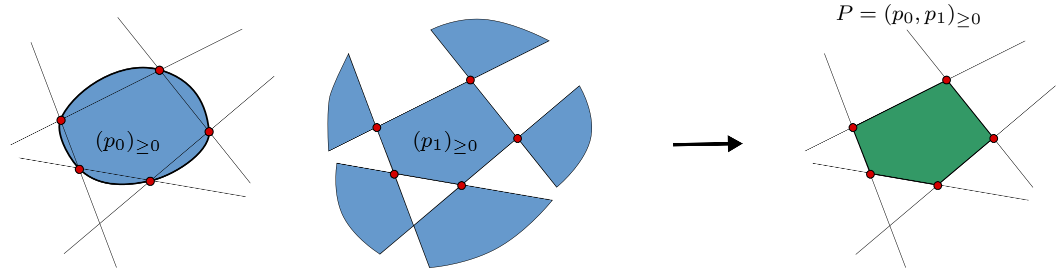

Figure 1. Representing a polygon by two polynomial inequalities

affine functions $q_1(x), \dots, q_m(x)$. We define $p_2(x) := q_1(x) \cdot \ldots \cdot q_m(x)$. Further on, we construct a strictly concave polynomial $p_0(x)$ vanishing on each vertex of P and such that the distance between P and $(p_0)_{\geq 0}$ is small enough; see also Fig. 2[1]. With each vertex v of P we associate a polynomial $b_v(x)$ such that $(b_v)_{\geq 0} = B(v) \cup \big(2\,v - B(v)\big)$, where $B(v)$ is a pointed convex cone with apex at v such that every edge of P incident with v lies in the boundary of $B(v)$; see Fig. 3. We define $p_1(x)$ as a "weighted combination" of the polynomials $b_v(x)$ with v ranging over the set of all vertices of P. More precisely, we set

$$p_1(x) := \sum_v f_v(x)^{2k}\, b_v(x)$$

where v ranges over all vertices of P, $k \in \mathbb{N}$ is suffiently large, and, for every vertex v of P, $f_v(x)$ is a polynomial given by

$$f_v(x) := \prod_{\substack{i=1,\dots,m \\ q_i(v) \neq 0}} q_i(x),$$

see also Fig. 4.

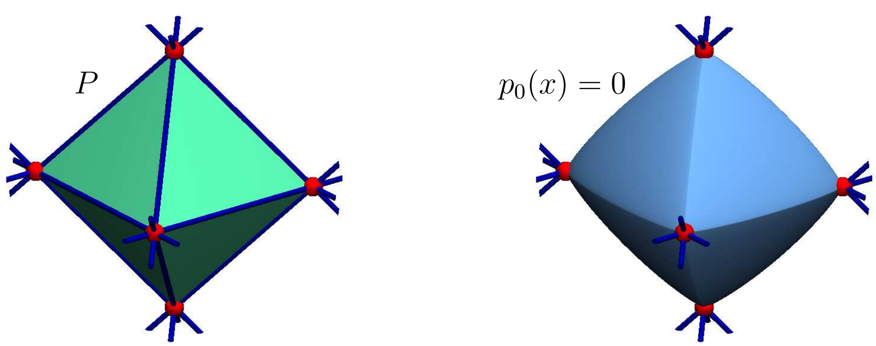

Figure 2. Regular octahedron and an approximation of the octahedron by a strictly concave polynomial surface

By construction $P \subseteq (p_0, p_1, p_2)_{\geq 0}$. We have $p_0(x) < 0$ for all x sufficiently far away from P. Further on, the region of all x satisfying $p_2(x) < 0$ is contiguous with all facets

[1]The figures in this subsection illustrate the construction for the case of the regular octahedron $\{(\xi_1, \xi_2, \xi_3) \in \mathbb{R}^3 : |\xi_1| + |\xi_2| + |\xi_3| \leq 1\}$. Since octahedra are not simple polytopes, they are not covered by the main result of [AH07] on representation of simple polytopes by d polynomial inequalities.

of P. Finally, if G is a vertex or an edge of P, x_0 is a point in the relative interior of G, and v and w are two vertices of P with $v \in G$ and $w \notin G$, respectively, then, for $x \to x_0$, the term $f_v(x)^{2k} b_v(x)$ dominates over $f_w(x)^{2k} b_w(x)$ provided k is sufficiently large. The latter observation is used for analysis of the properties of $(p_1)_{\geq 0}$.

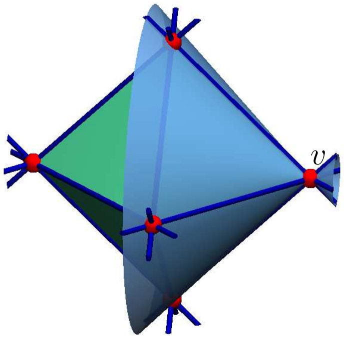

Figure 3. Polyhedron P and the surface given by the equation $b_v(x) = 0$

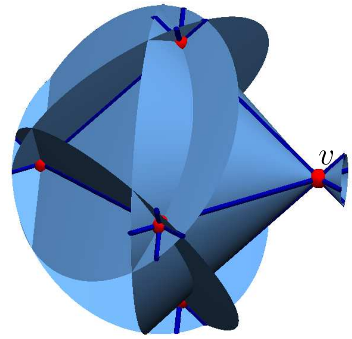

Figure 4. The surface given by the equation $f_v(x)^{2k} b_v(x) = 0$

The geometry behind the construction is illustrated by the diagram from Fig. 5, where $P_J := \{x \in \mathbb{R}^3 : p_j(x) \geq 0 \ \forall j \in J\}$ for $J \subseteq \{1, 2, 3\}$ with $J \neq \emptyset$ and an arrow between two sets indicates the inclusion relation.

2.3 The case of unbounded polyhedra

The case of unbounded d-polyhedra can be reduced to the case of bounded ones (at least for $d \leq 3$). In Subsection 7.2 we describe two possible ways of such a reduction. We sketch one of these ways below.

Assume that P is unbounded. We replace P by an isometric copy of P in $\mathbb{R}^{d+1}$ with $o \notin \operatorname{aff} P$. It suffices to restrict considerations to line-free polyhedra P. The union of the conical hull of P with the recession cone of P is denoted by $\operatorname{hom}(P)$. The set $\operatorname{hom}(P)$ is a pointed polyhedral cone. We consider a hyperplane H' in $\mathbb{R}^{d+1}$ such that $P' := H' \cap \operatorname{hom}(P)$ is bounded. Using the representations which were obtained above for the case of bounded polyhedra we find d polynomials $p_0(x), \ldots, p_{d-1}(x)$ such that

$$P' := \{x \in \operatorname{aff} P' : p_0(x) \geq 0, \ldots, p_{d-1}(x) \geq 0\}.$$

It turns out that the choice of $p_0(x), \ldots, p_{d-1}(x)$ can be "adjusted" so that $p_0(x), \ldots, p_{d-1}(x)$ become homogeneous polynomials of even degree, while the set $(p_0)_{\geq 0}$ becomes the union of two pointed convex cones which are symmetric to each other with respect to the origin and which intersect precisely at o. It follows that

$$P = \{x \in \operatorname{aff} P : p_0(x) \geq 0, \ldots, p_{d-1}(x) \geq 0\}.$$

3 Preliminaries from real algebraic geometry

For information on real algebraic geometry and, in particular, the geometry of semi-algebraic sets we refer to [ABR96], [BCR98]. If x is a variable in $\mathbb{R}^k$ ($k \in \mathbb{N}$), then $\mathbb{R}[x]$

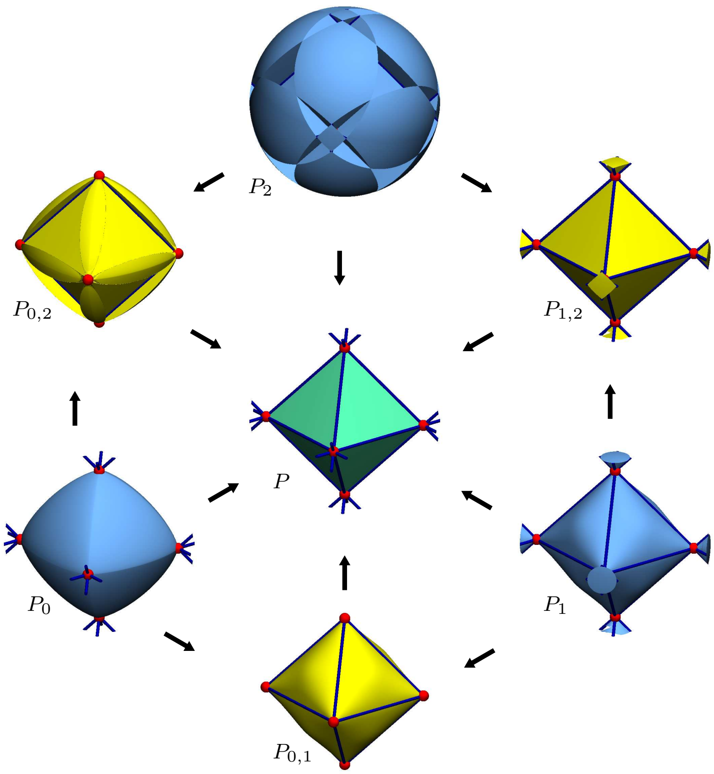

Figure 5. Diagram illustrating representation of a three-dimensional polytope by three polynomial inequalities.

stands for the class of k-variate polynomials over $\mathbb{R}$. Since $\mathbb{R}$ is a field of characteristic zero, we do not distinguish between real polynomials and real-valued polynomial functions. In particular, we can view affine functions as polynomials of degree at most one. A subset A of $\mathbb{R}^d$ is said to be *semi-algebraic* if

$$A = \bigcup_{i=1}^{n} \left\{ x \in \mathbb{R}^d : f_{i,1}(x) > 0, \ldots, f_{i,s_i}(x) > 0, \; g_i(x) = 0 \right\},$$

for some $n, s_1, \ldots, s_n \in \mathbb{N}$ and $f_{i,j}(x), \, g_i(x) \in \mathbb{R}[x]$ with $i \in \{1, \ldots, n\}$ and $j \in \{1, \ldots, s_i\}$.

A real valued function $f(x)$ defined on a semi-algebraic set $A \subseteq \mathbb{R}^d$ is said to be a *semi-algebraic function* if the graph of $f(x)$ is a semi-algebraic subset of $\mathbb{R}^{d+1}$. The following theorem presents *Łojasiewicz's Inequality*; see [Łoj59], [BCR98, Corollary 2.6.7].

Theorem 3.1. (Łojasiewicz 1959) *Let A be a bounded and closed semi-algebraic set in $\mathbb{R}^d$. Let $f(x)$ and $g(x)$ be continuous, semi-algebraic functions on A satisfying*

$$\{x \in A : f(x) = 0\} \subseteq \{x \in A : g(x) = 0\}.$$

Then there exist $n \in \mathbb{N}$ and $\lambda > 0$ such that

$$|g(x)|^n \leq \lambda \, |f(x)|$$

for every $x \in A$. □

An expression Φ is called a *first-order formula over* $\mathbb{R}$ if Φ is a formula built with a finite number of conjunctions, disjunctions, negations, and universal or existential quantifiers on variables, starting from formulas of the form $f(x) = 0$ or $f(x) > 0$ with $f \in \mathbb{R}[x]$; see [BCR98, Definition 2.2.3]. The *free variables* of Φ are those variables, which are not quantified. A formula with no free variables is called a *sentence*. Each sentence is either true or false. The following result is relevant for the constructive part of our main theorem; see [BPR06, Algorithm 12.30].

Theorem 3.2. (Tarski 1951, Seidenberg 1954) *Let Φ be a sentence over $\mathbb{R}$. Then there exists an algorithm that takes Φ and decides whether Φ is true or false.* □

Dealing with algorithmic statements like Theorem 3.2 (or Theorem 1.1), it assumed that a polynomial is given by its coefficients, a finite list of real coefficients occupies finite memory space, and arithmetic and comparison operations over real numbers are computable in one step.

4 Preliminaries from convexity

For information on convex bodies and polytopes we refer to [Sch93], [Gru07], [Zie95]. The cardinality of a set is denoted by $|\cdot|$. The origin, Euclidean norm, and scalar product in $\mathbb{R}^d$ are denoted by o, $\|\cdot\|$, and $\langle \cdot , \cdot \rangle$, respectively. We endow $\mathbb{R}^d$ with its Euclidean topology. By $\mathbb{B}^d(c, \rho)$ we denote the closed Euclidean ball in $\mathbb{R}^d$ with center at $c \in \mathbb{R}^d$ and radius $\rho > 0$. The notations aff, conv, int, and relint stand for the affine hull, convex hull, interior, and relative interior, respectively. If X and Y are non-empty sets in $\mathbb{R}^d$ we set $X \pm Y := \{x \pm y : x \in X,\ y \in Y\}$. The *Hausdorff distance* $\operatorname{dist}(X, Y)$ between non-empty, compact sets X, $Y \subseteq \mathbb{R}^d$ is defined by

$$\operatorname{dist}(X,Y) := \max\Big\{\max_{x\in X}\min_{y\in Y}\|x-y\|\,,\,\max_{y\in Y}\min_{x\in X}\|x-y\|\Big\},$$

see also [Sch93, p. 48]. It is known that $\operatorname{dist}(X, Y)$ is a metric on the space of non-empty, compacts sets in $\mathbb{R}^d$. The Hausdorff distance can be expressed by

$$\operatorname{dist}(X,Y) := \min\big\{\rho \geq 0 : X \subseteq Y + \mathbb{B}^d(o,\rho),\ Y \subseteq X + \mathbb{B}^d(o,\rho)\big\}\,. \tag{4.1}$$

If for a convex set $X \subseteq \mathbb{R}^d$ and a point $x_0 \in \mathbb{R}^d$ one has $x_0 + \alpha\, x \in X$ for every $\alpha \geq 0$ and $x \in X$, then X is said to be a *convex cone* with *apex* at x_0. Given a non-empty subset X of $\mathbb{R}^d$, we introduce the *conical hull* $\operatorname{cone}(X)$ of X as the set of all possible linear combinations $\lambda_1 x_1 + \cdots + \lambda_m x_m$ with $m \in \mathbb{N}$, $\lambda_i \geq 0$, and $x_i \in X$ for $i \in \{1, \ldots, m\}$. Clearly, $\operatorname{cone}(X)$ is a convex cone with apex at the origin.

The notions of polyhedron and polytope were defined in the introduction. Polyhedra (resp. polytopes) of dimension n are referred to as *n-polyhedra* (resp. *n-polytopes*). A subset of $\mathbb{R}^d$ which is both a cone and polyhedron is said to be a *polyhedral cone*. A polyhedron is said to be *line-free* if it does not contain a straight line. The set $\operatorname{rec}(P) := \big\{u \in \mathbb{R}^d : P + u \subseteq P\big\}$ is said to be the *recession cone* of P. A line-free polyhedral cone is said to be *pointed*. Every polyhedron P can be represented by

$$P = Q + L + C, \tag{4.2}$$

where Q is a polytope, L is a linear subspace of $\mathbb{R}^d$, and C is a pointed polyhedral cone; see [Zie95, Section 1.5]. For (4.2) one necessarily has $L + C = \mathrm{rec}(P)$. If $o \notin \mathrm{aff}\, P$, we introduce

$$\mathrm{hom}(P) := \mathrm{cone}(P) \cup \mathrm{rec}(P).$$

It is known that $\mathrm{hom}(P)$ is a polyhedral cone; see [Zie95, pp. 44-45]. Furthermore, it can be verified that if P is line-free, then $\mathrm{hom}(P)$ is pointed.

For a convex polytope P in $\mathbb{R}^d$ we introduce the *support function* and *exposed facet* in direction u by the equalities

$$h(P, u) := \max \{\langle x, u\rangle : x \in P\}$$

and

$$F(P, u) := \{x \in P : \langle x, u\rangle = h(P, u)\},$$

respectively. Given a polytope P in $\mathbb{R}^d$, $\mathcal{F}(P)$ denotes the class of all faces of P, and, for $i \in \{-1, \dots, d\}$, $\mathcal{F}_i(P)$ stands for the class of all *i-faces* of P (i.e., faces of dimension i). We recall that for every point x of P there exists a unique non-empty face G of P with $x \in \mathrm{relint}\, P$. Consequently, P is the disjoint union of the relative interiors of all non-empty faces of P. A face G of an n-polytope P in $\mathbb{R}^d$ is said to be a *facet* of P if G has dimension $n - 1$. If G is a face of P, we define

$$\mathcal{F}_i(G, P) := \{F \in \mathcal{F}_i(P) : G \subseteq F\}.$$

By $\mathrm{vert}(P)$ we denote the set of all *vertices* of P (i.e., the set of 0-dimensional faces). If F is a facet of P by $u_F(P)$ we denote the unit normal of P which is parallel to $\mathrm{aff}(P)$ and satisfies $F(P, u_F(P)) = F$. More generally, if G is a proper face of P, we define

$$u_G(P) := \Bigg(\sum_{F\in\mathcal{F}_{n-1}(G,P)} u_F(P)\Bigg) \Bigg/ \Bigg\|\sum_{F\in\mathcal{F}_{n-1}(G,P)} u_F(P)\Bigg\|,$$

where $n := \dim(P)$. It is not hard to see that $F(P, u_G(P)) = G$.

With every facet F of P we associate the affine function

$$q_F(P, x) := \frac{h(P, u_F(P)) - \langle u_F(P), x\rangle}{\mathrm{diam}(P)},$$

where

$$\mathrm{diam}(P) := \max \{\|x - y\| : x,\, y \in P\} = \max \{\|x - y\| : x,\, y \in \mathrm{vert}(P)\}.$$

By construction, $0 \le q_F(P, x) < 1$ for every $x \in P$ with equality $q_F(P, x) = 0$ if and only if $x \in F$.

For $v \in \mathrm{vert}(P)$, the *supporting cone* $S(P, v)$ of P at v is given by

$$S(P, v) := \mathrm{cone}(P - v),$$

see also [Sch93, p. 70]. If $\dim P = d$, then

$$S(P, v) = \big\{x \in \mathbb{R}^d : \big\langle u_F(P), x\big\rangle \le 0 \;\forall\, F \in \mathcal{F}_{d-1}(v, P)\big\}$$

and by this

$$v+S(P,v)=\left\{x\in\mathbb{R}^d : q_F(P,x)\le 0\ \forall\, F\in\mathcal{F}_{d-1}(v,P)\right\}.$$

For every $v\in\operatorname{vert}(P)$ we define the hyperplane

$$H_v(P):=\left\{x\in\mathbb{R}^d : \left\langle x,u_v(P)\right\rangle=-1\right\}$$

and the polytope

$$P_v:=S(P,v)\cap H_v(P).$$

It can be shown that $\operatorname{cone}(P_v)=S(P,v)$. Hence the cone

$$S_\rho(P,v):=\operatorname{cone}\Big(\big(P_v+\mathbb{B}^d(o,\rho)\big)\cap H_v(P)\Big).$$

with $\rho>0$ and $v\in\operatorname{vert}(P)$ can be viewed as an "outer approximation" of the supporting cone with the parameter ρ controlling the quality of the approximation. For every $v\in\operatorname{vert}(P)$ one has

$$P\setminus\{v\}\subseteq\operatorname{int}\big(v+S_\rho(P,v)\big). \tag{4.3}$$

For $x\in\mathbb{R}^d$ and a polytope P in $\mathbb{R}^d$ and $x\in\operatorname{aff}P$ we introduce the notation

$$\mathcal{F}^-(P,x):=\{F\in\mathcal{F}_{d-1}(P) : q_F(P,x)\le 0\}.$$

The class $\mathcal{F}^-(P,x)$ be interpreted as the set of all facets of the polytope P which are visible from x.

In Theorem 1.1 and the remaining statements dealing with algorithms a polyhedron is assumed to be given by a system of affine inequalities (the so-called *H-representation*) and a polynomial by a list of its coefficients. It is not difficult to show that there exists an algorithm that takes a polytope P and constructs all functions $q_F(P,x)$ where F is a facet of P.

5 A family of neighborhoods of a 3-polytope

Let $\varepsilon>0$, $\rho>0$, and P be a 3-polytope in $\mathbb{R}^3$. We set $U_P(P,\varepsilon,\rho):=P$ and $U_F(P,\varepsilon,\rho):=\left\{x\in\mathbb{R}^3 : \mathcal{F}^-(P,x)=\{F\}\right\}$ for $F\in\mathcal{F}_2(P)$. Furthermore, for $I\in\mathcal{F}_1(P)$ and $v\in\operatorname{vert}(P)$ we define

$$\begin{aligned} U_I(P,\varepsilon,\rho) &:= \big(I+\mathbb{B}^3(o,\varepsilon)\big)\cap\left\{x\in\mathbb{R}^3 : \mathcal{F}_2(I,P)\subseteq\mathcal{F}^-(P,x)\right\} \\ &\quad\cap\Bigg(\bigcap_{v\in\operatorname{vert}(P)} S_\rho(P,v)\Bigg), && (5.1)\\ U_v(P,\varepsilon,\rho) &:= \mathbb{B}^3(v,\varepsilon)\setminus\big(v+\operatorname{int}S_\rho(P,v)), && (5.2)\end{aligned}$$

see also Figs. 6 and 7. We also introduce the region

$$U_v'(P,\varepsilon,\rho):=\mathbb{B}^3(v,\varepsilon)\setminus\Big((v+\operatorname{int}S_\rho(P,v))\cup(v-\operatorname{int}S_\rho(P,v))\Big),$$

which is a subset of $U_v(P,\varepsilon,\rho)$; see Fig. 8. With P we associate the set $U(P,\varepsilon,\rho)$

$$U(P,\varepsilon,\rho):=\bigcup_{G\in\mathcal{F}(P)\setminus\{\emptyset\}} U_G(P,\varepsilon,\rho),$$

The main statement of this section is Propsition 5.1. The statement of this proposition is intuitively clear. However, we are not aware of any short proof of it.

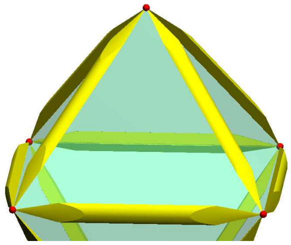

Figure 6. The regions $U_I(P, \varepsilon, \rho)$ with $I \in \mathcal{F}_1(P)$

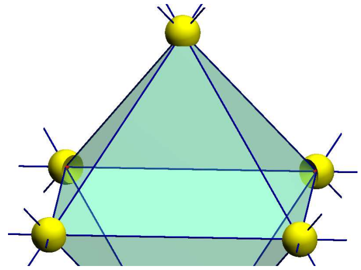

Figure 7. The regions $U_v(P, \varepsilon, \rho)$ with $v \in \mathrm{vert}(P)$

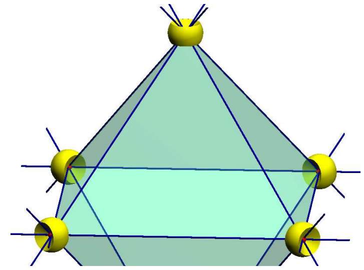

Figure 8. The regions $U'_v(P, \varepsilon, \rho)$ with $v \in \mathrm{vert}(P)$

Proposition 5.1. *Let P be a 3-polytope in $\mathbb{R}^3$. Then $P \subseteq \mathrm{int}\, U(P, \varepsilon, \rho)$ for all sufficiently small $\rho > 0$ and all $\varepsilon > 0$.* $\square$

The proof of Proposition 5.1 is based on a number of auxiliary statements. The proofs of the simple Lemmas 5.2, 5.3, and 5.4 are omitted.

Lemma 5.2. *Let P be a d-polytope in $\mathbb{R}^d$, G a non-empty face of P, and $x \in \mathrm{relint}\, G$. Then there exists $\varepsilon > 0$ such that every $y \in \mathbb{B}^d(x, \varepsilon)$ satisfies $\mathcal{F}^-(P, y) \subseteq \mathcal{F}_{d-1}(G, P)$* $\square$

If a point x outside P is sufficiently close to P, then all facets of P visible from x have a vertex in common. This is stated in a more formal way in the following lemma.

Lemma 5.3. *Let P be a d-polytope in $\mathbb{R}^d$. Then for some $\varepsilon > 0$ and every $x \in P + \mathbb{B}^d(o, \varepsilon)$ there exists $v \in \mathrm{vert}(P)$ satisfying $\mathcal{F}^-(P, x) \subseteq \mathcal{F}_{d-1}(v, P)$.* $\square$

Lemma 5.4. *Let P be a d-polytope in $\mathbb{R}^d$ with $o \in \mathrm{vert}(P)$ and let $F \in \mathcal{F}_{d-1}(o, P)$. Then $G := \mathrm{aff}\, F \cap P_o$ is a facet of P_o. Furthermore, for some $\alpha > 0$ one has*

$$q_F(P, x) = \alpha \cdot q_G(P_o, x) \qquad \forall\, x \in H_o(P).$$

$\square$

Lemma 5.5. *Let P be a 3-polytope in $\mathbb{R}^3$. Then there exists $\rho > 0$ such that for all $v \in \mathrm{vert}(P)$ and $x \in \big(v + S_\rho(P, v)\big) \setminus \{v\}$ one can find $I \in \mathcal{F}_1(v, P)$ satisfying*

$$\mathcal{F}^-(P, x) \cap \mathcal{F}_2(v, P) \subseteq \mathcal{F}_2(I, P). \tag{5.3}$$

Proof. Consider an arbitrary $v \in \mathrm{vert}(P)$. Replacing P by its appropriate translation we assume that $v = o$. By Lemma 5.3 applied to P_o, one can choose $\rho > 0$ such that for every $x \in H_o(P) \cap (P_o + \mathbb{B}^3(o, \rho))$ there exists $w \in \mathrm{vert}(P_o)$ satisfying $\mathcal{F}^-(P_o, x) \subseteq \mathcal{F}_1(w, P_o)$. Take an arbitrary $x \in S_\rho(P, o) \setminus \{o\}$. Then $\langle x, u_o(P) \rangle < 0$, and we introduce

$$y := -\frac{x}{\langle x, u_o(P) \rangle} \in H_o(P) \cap \big(P_o + \mathbb{B}^3(o, \rho)\big).$$

By the choice of ρ, there exists $w \in \mathrm{vert}(P_v)$ with $\mathcal{F}^-(P_o, y) \subseteq \mathcal{F}_1(w, P_o)$. In view of Lemma 5.4, the latter inclusion implies $\mathcal{F}^-(P, x) \cap \mathcal{F}_2(o, P) \subseteq \mathcal{F}_2(I, P)$, where $I \in \mathcal{F}_1(o, P)$ is such that $H_o(P) \cap \mathrm{cone}(I) = \{w\}$. This shows (5.3). $\square$

Now we are ready to prove the main statement of this section.

Proof of Proposition 5.1. We pick an arbitrary $x \in P$ and show that $x \in \operatorname{int} U(P, \varepsilon, \rho)$. By G we denote the unique face of P with $x \in \operatorname{relint} G$. If $G = P$, then $x \in \operatorname{int} P$ and by this $x \in \operatorname{int} U(P, \varepsilon, \rho)$. If G is a facet of P, then, in view of Lemma 5.2, all points sufficiently close to x lie in $P \cup U_G(P, \varepsilon, \rho)$. If G is an edge of P, we show that

$$x \in \operatorname{int}\left(P \cup \left(\bigcup_{F \in \mathcal{F}_2(G,P)} U_F(P, \varepsilon, \rho)\right) \cup U_G(P, \varepsilon, \rho)\right). \tag{5.4}$$

In fact, it is easy to see that for every $v \in \operatorname{vert}(P)$ one has $x \in v + \operatorname{int} S_\rho(P, v)$ and $x \in \operatorname{int}(G + \mathbb{B}^3(o, \varepsilon))$. Furthermore, applying Lemma 5.2 we obtain

$$x \in \operatorname{int}\left\{y \in \mathbb{R}^3 : \mathcal{F}^-(F, y) \subseteq \mathcal{F}_2(G, P)\right\}.$$

Thus, (5.4) is fulfilled, and by this $x \in \operatorname{int} U(P, \varepsilon, \rho)$.

It remains to consider the case $x = v$ for some $v \in \operatorname{vert}(P)$. We assume that ρ is small enough so that the assertion of Lemma 5.5 is fulfilled. We show that $x \in \operatorname{int} U(P, \varepsilon, \rho)$ by contradiction. Assume that there exists a sequence $(x_n)_{n=1}^{\infty}$ converging to x and such that $x_n \notin U(P, \varepsilon, \rho)$ for every $n \in \mathbb{N}$. Take an arbitrary $\mathcal{F}^* \subseteq \mathcal{F}_2(P)$ such that $\mathbb{N}^* := \{n \in \mathbb{N} : \mathcal{F}^-(P, x_n) = \mathcal{F}^*\}$ is infinite. By Lemma 5.2, for all sufficiently large $n \in \mathbb{N}$ one has $\mathcal{F}^-(P, x_n) \subseteq \mathcal{F}_2(v, P)$. Hence $\mathcal{F}^* \subseteq \mathcal{F}_2(v, P)$. If $|\mathcal{F}^*| = 1$, then $x_n \in U(P, \varepsilon, \rho)$ for every $n \in \mathbb{N}^*$. If $|\mathcal{F}^*| \ge 3$, then for all sufficiently large n one has $x_n \notin v + S_\rho(P, v)$ and by this $x_n \in \mathbb{B}^3(v, \varepsilon) \setminus (v + \operatorname{int} S_\rho(P, v)) = U_v(\varepsilon, \rho)$. In fact, if we assume the contrary, i.e., $x_n \in v + S_\rho(P, v)$ for all sufficiently large $n \in \mathbb{N}^*$, then, by (5.3), we see that for some $I \in \mathcal{F}_1(v, P)$ one has $\mathcal{F}^* \cap \mathcal{F}_2(v, P) \subseteq \mathcal{F}_2(I, P)$. In view of $\mathcal{F}^* \subseteq \mathcal{F}_2(v, P)$, the latter implies $\mathcal{F}^* \subseteq \mathcal{F}_2(I, P)$. Hence $|\mathcal{F}^*| \le 2$, a contradiction. We consider the remaining case $|\mathcal{F}^*| = 2$. In view of (4.3), one has $x_n \in S_\rho(P, w)$ for all sufficiently large n and every $w \in \operatorname{vert}(P) \setminus \{v\}$. It is easy to see that for all sufficiently large n one has $x_n \in \mathbb{B}^3(v, \varepsilon)$. Hence, for all large n one has $x_n \in U_v(P, \varepsilon, \rho)$ if $x \notin v + \operatorname{int} S_\rho(P, v)$ and, taking into account the assertion of Lemma the assertion of Lemma 5.5 and the definition of $U_I(P, \varepsilon, \rho)$, one has $x_n \in U_I(P, \varepsilon, \rho)$ for some $I \in \mathcal{F}_1(v, P)$ provided $x \in v + S_\rho(P, v)$. Consequently for all large n one has $x_n \in U(P, \varepsilon, \rho)$. This yields the assertion of the proposition. $\square$

6 Approximation of polytopes and polyhedral cones

A polynomial $f \in \mathbb{R}[x]$ of degree n is said to be *homogeneous* if $f(\alpha\, x) = \alpha^n f(x)$ for all $\alpha \in \mathbb{R}$ and $x \in \mathbb{R}^d$. Consider a hyperplane H in $\mathbb{R}^d$ with $o \notin H$. Obviously, H can be represented by

$$H = \left\{x \in \mathbb{R}^d : \langle x, u \rangle = 1\right\} \tag{6.1}$$

with $u \in \mathbb{R}^d \setminus \{o\}$ uniquely determined by H. Consider a polynomial $f \in \mathbb{R}[x]$ of degree n. The *homogeneous continuation* of $f(x)|_H$ (i.e., the restriction of $f(x)$ to H) to $\mathbb{R}^d$ is defined as the polynomial $\tilde{f} \in \mathbb{R}[x]$ uniquely determined by $\tilde{f}(x) = \langle x, u \rangle^n f\big(x / \langle x, u \rangle\big)$ with $x \in \mathbb{R}^d \setminus H$. Clearly, $\tilde{f}$ is a homogeneous polynomial.

We say that there is an algorithm that constructs a sequence $\big(p_l(x)\big)_{l=1}^{\infty}$ of polynomials from the given input if there exists an algorithm which constructs $p_l(x)$ from $l \in \mathbb{N}$ and the given input. Theorem 6.1 and Proposition 6.3 are the main statements of this section.

The former is concerned with a special type of approximation of polytopes and the latter with approximation of supporting cones of polytopes. Given a polytope P in $\mathbb{R}^d$, we introduce the following conditions on a sequence $\big(g_l(x)\big)_{l=1}^{\infty}$ of polynomials from $\mathbb{R}[x]$.

$\mathcal{A}(P)$: The Hausdorff distance from P to $\{x \in \operatorname{aff} P : g_l(x) \geq 0\}$ converges to zero, as $l \to \infty$.

$\mathcal{I}(P)$: For every $v \in \operatorname{vert}(P)$ and $l \in \mathbb{N}$ one has $g_l(v) = 0$.

$\mathcal{C}(P)$: For every $l \in \mathbb{N}$ the function $g_l(x)|_{\operatorname{aff} P}$ is strictly concave.

$\mathcal{H}(P)$: For every $l \in \mathbb{N}$ one has

$$\big\{x \in \mathbb{R}^d : \tilde{g}_l(x) \geq 0,\ \langle x, u\rangle \geq 0\big\} = \operatorname{cone}\big(\{x \in \operatorname{aff} P : g_l(x) \geq 0\}\big),$$

where $\tilde{g}_l(x)$ is a homogeneous continuation of $g_l(x)|_{\operatorname{aff} P}$.

The notations $\mathcal{A}$, $\mathcal{I}$, $\mathcal{C}$, and $\mathcal{H}$ are derived from the words 'approximation', 'interpolation', 'concavity', and 'homogeneity', respectively. Condition $\mathcal{H}(P)$ makes sense only for the case when $o \notin \operatorname{aff} P$.

Theorem 6.1. *Let P be a polytope in $\mathbb{R}^d$. Then the following statements hold true.*

I. There exists an algorithm that takes P and constructs a sequence $(g_l(x))_{l=1}^{\infty}$ of polynomials from $\mathbb{R}[x]$ that satisfy $\mathcal{A}(P)$, $\mathcal{I}(P)$, and $\mathcal{C}(P)$.

II. For every sequence $\big(g_l(x)\big)_{l=1}^{\infty}$ satisfying $\mathcal{A}(P)$, $\mathcal{I}(P)$, and $\mathcal{C}(P)$ there exists $\rho > 0$ with

$$\big(v - S_\rho(P, v)\big) \cap \{x \in \operatorname{aff} P : g_l(P, x) \geq 0\} = \{v\} \qquad \forall\, v \in \operatorname{vert}(P)\ \forall\, l \in \mathbb{N}. \tag{6.2}$$

III. If $\dim(P) = d-1$ and $o \notin \operatorname{aff} P$, there exists an algorithm that takes P and constructs a sequence $\big(g_l(x)\big)_{l=1}^{\infty}$ of polynomials from $\mathbb{R}[x]$ that satisfy $\mathcal{A}(P)$, $\mathcal{I}(P)$, $\mathcal{C}(P)$, and $\mathcal{H}(P)$.

Proof. Let $n := \dim(P)$.

Part I. For $l \in \mathbb{N}$ we define $g_l(x)$ by

$$g_l(x) := 1 - \sum_{v \in \operatorname{vert}(P)} y_{v,l} \left(\frac{1}{|\mathcal{F}_{n-1}(v, P)|} \sum_{F \in \mathcal{F}_{n-1}(v,P)} \big(1 - q_F(P, x)\big)^{2(l+l_0)} \right)^{2(l+l_0)} \tag{6.3}$$

where $l_0 \in \mathbb{N}$ and $y_{v,l} \in \mathbb{R}$. In [AH07] it was shown that an appropriate $l_0 \in \mathbb{N}$ and scalars $y_{v,l}$ with $v \in \operatorname{vert}(P)$ and $l \in \mathbb{N}$ can be constructed such that the sequence $\big(g_l(x)\big)_{l=1}^{\infty}$ satisfies $\mathcal{I}(P)$ an $\mathcal{A}(P)$. By construction, $\big(g_l(x)\big)_{l=1}^{\infty}$ also satisfies the condition $\mathcal{C}(P)$.

Part II. Without loss of generality we may assume that $n = d$. Consider an arbitrary $v \in \operatorname{vert}(P)$. Replacing P by an appropriate translation we assume that $v = o$. Let $\rho'_o := \min\{\|x - y\| : x \in P, y \in (-P_o)\}$. We show that there exists $l'_o \in \mathbb{N}$ such that one has

$$\forall\, l \geq l'_o\ \forall\, x \in \big(-P_o + \mathbb{B}^d(o, \rho'_o/2)\big) \cap (-H_o(P))\ :\ g_l(x) < 0. \tag{6.4}$$

Assume the contrary. Then there exist sequences $(l_k)_{k=1}^{\infty}$, $(x_k)_{k=1}^{\infty}$, and $(y_k)_{k=1}^{\infty}$ such that $l_k \in \mathbb{N}$, $x_k \in (-H_o(P))$, $y_k \in -P_o$, $\|x_k - y_k\| \le \rho'_o/2$, and $g_{l_k}(x_k) \ge 0$ for every $k \in \mathbb{N}$. Then, for every $k \in \mathbb{N}$, we obtain

$$\begin{aligned}\operatorname{dist}\big(P, \{x \in \operatorname{aff} P : g_{l_k}(x) \ge 0\}\big) &\ge \min_{x \in P} \|x - x_k\| \\ &\ge \min_{x \in P} \|x - y_k\| - \|x_k - y_k\| \\ &\ge \rho'_o - \|x_k - y_k\| \ge \rho'_o/2 > 0,\end{aligned}$$

a contradiction to the assumption that $\big(g_l(x)\big)_{l=1}^{\infty}$ satisfies $\mathcal{A}(P)$. Let us fix l'_o satisfying (6.4). Next we show that for every for every $l \in \mathbb{N}$ and every $x \in -P_o$ one has $g_l(x) < 0$. Assume the contrary, i.e. $g_l(x) \ge 0$. Since $\big(g_l(x)\big)_{l=1}^{\infty}$ satisfies $\mathcal{C}(P)$ and $\mathcal{I}(P)$ we deduce that $g_l(y) > 0$ for every $y \in P \setminus \operatorname{vert}(P)$. For a sufficiently small $\varepsilon > 0$ one has $-\varepsilon\, x \in P \setminus \operatorname{vert}(P)$. Then o is a convex combination of $-\varepsilon\, x$ and y (with non-zero coefficients). Hence, from strict convexity of $g_l(x)$ and $g_l(-\varepsilon\, x) > 0$, $g_l(x) \ge 0$, we deduce that $g_l(o) > 0$, a contradiction. Hence $g_l(x) < 0$ for every $l \in \mathbb{N}$ and every $x \in -P_o$ and therefore we may fix $\rho_{o,l} > 0$ such that $\big(-P_o + \mathbb{B}^d(o, \rho'_{o,l})\big) \cap (-H_o)$ is disjoint with $\{x \in \operatorname{aff} P : g_{l_k}(x) \ge 0\}$. Applying the above arguments to all vertices of P we determine quantities $\rho_{v,l}$ and ρ'_v with $v \in \operatorname{vert}(P)$ and $l \in \mathbb{N}$.

The assertion follows by setting

$$\begin{aligned} l' &:= \max\{l'_v : v \in \operatorname{vert}(P)\}, \\ \rho_v &:= \min\{\rho_{v,1}, \dots, \rho_{v,l'}, \rho'_v/2\}\end{aligned}$$

for every $v \in \operatorname{vert}(P)$ and taking $\rho := \min\{\rho_v : v \in \operatorname{vert}(P)\}$.

Part III. We assume $o \notin \operatorname{aff} P$. Without loss of generality we also assume that $n = d-1$. For $l \in \mathbb{N}$ the homogeneous continuation $\tilde{g}_l(x)$ of $g_l(x)|_{\operatorname{aff} P}$ can be expressed by

$$\tilde{g}_l(x) := \\ \langle x, u\rangle^{4(l+l_0)^2} - \sum_{v \in \operatorname{vert}(P)} y_{v,l} \left(\frac{1}{|\mathcal{F}_{d-2}(v,P)|} \sum_{F \in \mathcal{F}_{d-2}(v,P)} \Big(\langle x, u\rangle - \tilde{q}_F(P,x)\Big)^{2(l+l_0)} \right)^{2(l+l_0)}, \tag{6.5}$$

where $\tilde{q}_F(P,x)$ is a homogeneous continuation of $q_F(P,x)|_{\operatorname{aff} P}$, i.e.

$$\tilde{q}_F(P,x) := \frac{h(P, u_F(P))\,\langle u, x\rangle - \langle u_F(P), x\rangle}{\operatorname{diam}(P)},$$

If $\langle x, u\rangle = 0$, then (6.5) amounts to

$$\tilde{g}_l(x) := -\sum_{v \in \operatorname{vert}(P)} y_{v,k} \left(\frac{1}{|\mathcal{F}_{d-2}(v,P)|} \sum_{F \in \mathcal{F}_{d-2}(v,P)} \left(\frac{\langle u_F(P), x\rangle}{\operatorname{diam}(P)}\right)^{2(l+l_0)} \right)^{2(l+l_0)}. \tag{6.6}$$

Thus, if $\langle x, u\rangle = 0$, we have $\tilde{g}_l(x) \le 0$ with equality if and only if $x = o$. Directly from the definition of homogeneous continuation it follows

$$\big\{x \in \mathbb{R}^d : \langle x, u\rangle > 0,\ \tilde{g}_l(x) \ge 0\big\} \cup \{o\} = \operatorname{cone}\big(\{x \in H : g_l(x) \ge 0\}\big). \tag{6.7}$$

Equalities (6.6) and (6.7) yield the assertion. $\square$

We shall need the following observation.

Lemma 6.2. *The degree of every non-zero concave polynomial over $\mathbb{R}[x]$ is even.* $\square$

Proposition 6.3. *Let P be a 3-polytope in $\mathbb{R}^3$ and let $v \in \operatorname{vert}(P)$. Then there exists an algorithm which takes P and constructs a sequence of polynomials $\big(b_{v,l}(x)\big)_{l=1}^{\infty}$ satisfying the following conditions:*

I. For every $l \in \mathbb{N}$ the polynomial $b_{v,l}(x)$ is of even degree

II. For every $l \in \mathbb{N}$ the polynomial $b_{v,l}(x+v)$ is homogeneous.

III. For all $\rho > 0$, $l \in \mathbb{N}$, $I \in \mathcal{F}_1(v,P)$, and $x \in v + S_\rho(P,v)$ satisfying $\mathcal{F}^-(P,x) = \mathcal{F}_2(I,P)$ one has $b_{v,l}(x) \le 0$ with equality if and only if $x \in \operatorname{cone}(I-v)+v$.

IV. For every $l \in \mathbb{N}$ the set

$$B_l(P,v) := \big\{x \in \mathbb{R}^3 : \big\langle x-v, u_v(P)\big\rangle \le 0,\ b_{v,l}(x) \ge 0\big\} \tag{6.8}$$

is a convex cone which has apex at v, which satisfies $P \subseteq B_l(P,v)$.

V. For every $\rho > 0$

$$(v+S(P,v)) \cup (v-S(P,v)) \subseteq (b_{v,l})_{\ge 0} \subseteq \{v\} \cup (v+\operatorname{int} S_\rho(P,v)) \cup (v-\operatorname{int} S_\rho(P,v))$$

if $l \in \mathbb{N}$ is sufficiently large.

$\square$

Proof. Replacing P by an appropriate translation we may assume that $v = o$. Applying Theorem 6.1(III), we determine a sequence $\big(g_{o,l}(x)\big)_{l=1}^{\infty}$ which satisfies $\mathcal{A}(P_o)$, $\mathcal{I}(P_o)$, and $\mathcal{C}(P_o)$, respectively. We define $b_{o,l}(x) := \tilde{g}_{o,l}(x-v)$, where $\tilde{g}_{o,l}(x)$ is the homogeneous continuation of $g_{o,l}(x)|_{H_o(P)}$.

The part of Statement I follows Lemma 6.2. Statements II follows by construction.

Let us show Statement III. Consider an arbitrary $\rho > 0$ and $x \in S_\rho(P,v)$ with $\mathcal{F}^-(P,x) = \mathcal{F}_2(I,P)$. From the definition of $S_\rho(P,v)$ it follows that $\langle x, u_o(P)\rangle < 0$ and by this $y := \frac{x}{|\langle x, u_o(P)\rangle|} \in H_o(P)$. The intersection of $\operatorname{cone}(I)$ and $H_o(P)$ is a vertex of P_o, which we denote by w. Since $\mathcal{F}^-(P,x) = \mathcal{F}_2(I,P)$, taking into account Lemma 5.4, we obtain $\mathcal{F}^-(P_o,y) = \mathcal{F}_2(w,P_o)$. Consequently, $y \in w - S(P_o,w)$. Then, by Theorem 6.1(II), we get $g_{o,l}(y) \le 0$ with equality if and only if $y = w$. Consequently, $\tilde{g}_{o,l}(x) \le 0$ with equality if and only if $x \in S_\rho(P,o) \cap \operatorname{aff} I = \operatorname{cone}(I)$.

Statement IV follows directly from Theorem 6.1(III).

It remains to show Statement V. Consider an arbitrary $\rho > 0$. Since $(g_{o,l}(x))_{l=1}^{\infty}$ satisfies $\mathcal{I}(P_o)$ and $\mathcal{C}(P_o)$ and in view of the definition of $b_{o,l}(x)$ we obtain

$$S(P,o) \cup (-S(P,o)) \subseteq (b_{o,l})_{\ge 0}.$$

By construction, $b_{o,l}(x) = b_{o,l}(-x)$ for every $x \in \mathbb{R}^3$. Consequently $(b_{o,l})_{\ge 0} = B_l(P,o) \cup (-B_l(P,o))$. Since $\big(g_{o,l}(x)\big)_{l=1}^{\infty}$ satisfies $\mathcal{A}(P_o)$, $\mathcal{I}(P_o)$, and $\mathcal{C}(P_o)$, if l is large enough the relation

$$(b_{o,l})_{\ge 0} \subseteq S_{\rho/2}(P,o) \cup (-S_{\rho/2}(P,o))$$

holds true. Obviously

$$S_{\rho/2}(P,o) \cup (-S_{\rho/2}(P,o)) \subseteq \{o\} \cup \operatorname{int} S_\rho(P,o) \cup (-\operatorname{int} S_\rho(P,o)),$$

and we arrive at (6.8). $\square$

7 Proof of the main result

7.1 The case of bounded polyhedra

A statement similar to the following proposition was shown in [Ber98, Section 3.2]; see also the survey [Hen07].

Proposition 7.1. *Let P be a convex polygon in $\mathbb{R}^2$, $p_0(x)$ be a strictly concave polynomial vanishing on each vertex of P, and $p_1(x) := \prod_{I \in \mathcal{F}_1(P)} q_I(P, x)$. Then $P = (p_0, p_1)_{\geq 0}$.*

Proof. The inclusion $P \subseteq (p_0, p_1)_{\geq 0}$ is trivial. For proving the reverse inclusion we fix an arbitrary $x_0 \in \mathbb{R}^2 \setminus P$. We distinguish the following three cases.

Case 1: $p_1(x_0) < 0$. Obviously $x_0 \notin (p_0, p_1)_{\geq 0}$.

Case 2: $p_1(x_0) = 0$. There exists $I \in \mathcal{F}_1(P)$ with $q_F(P, x_0) = 0$. The polynomial $p_0(x)$ is strictly concave on aff I, non-negative on I, and equal zero at the endpoints of I. The latter easily implies that $p_0(x_0) < 0$. Hence $x_0 \notin (p_0, p_1)_{\geq 0}$.

Case 3: $p_1(x_0) > 0$. Then $q_{I_1}(P, x_0) < 0$ and $q_{I_2}(P, x_0) < 0$ for two distinct edges I_1, I_2 of P. The edges I_1 and I_2 are not parallel. The intersection point y of aff I_1 and aff I_2 satisfies $q_{I_1}(P, y) = q_{I_2}(P, y) = 0$. We fix points x_1 and x_2 belonging to I_1 and I_2, respectively, and not coinciding with y. By construction, the point y lies in the interior of the triangle conv$\{x_0, x_1, x_2\}$, that is, $y = \lambda_0 x_0 + \lambda_1 x_1 + \lambda_2 x_2$ for some λ_0, λ_1, $\lambda_2 > 0$ with $\lambda_0 + \lambda_1 + \lambda_2 = 1$. We show by contradiction, that $p_0(x_0) < 0$. Assume the contrary. Then, by the strict concavity of $p_0(x)$, we obtain

$$p_0(y) > \sum_{j=0}^{2} \lambda_j \, p_0(x_j) \geq 0.$$

The inequality $p_0(y) > 0$ contradicts the conclusion made in Case 2 and applied to the point y in place of x_0. Hence $p_0(x_0) < 0$ and by this $x_0 \notin (p_0, p_1)_{\geq 0}$. □

The following theorem implies Theorem 1.1 for the case when P is a 3-polytope.

Theorem 7.2. *Let $d \in \{2, 3\}$ and P be a d-dimensional polytope in $\mathbb{R}^d$. Let $(g_m(x))_{m=1}^{\infty}$ be a sequence of polynomials satisfying conditions $\mathcal{A}(P)$, $\mathcal{I}(P)$, and $\mathcal{C}(P)$. Then there exists an algorithm that takes P and constructs polynomials $p_1(x), \ldots, p_{d-1}(x) \in \mathbb{R}[x]$ such that $P = (p_0, \ldots, p_{d-1})_{\geq 0}$ and*

$$p_0(x) = g_m(x). \tag{7.1}$$

for some $m \in \mathbb{N}$.

Proof. The case $d = 2$ follows directly from Theorem 6.1 and Proposition 7.1. We consider the case $d = 3$. First we show the existence of polynomials $p_0(x)$, $p_1(x)$, $p_2(x)$ satisfying $P = (p_0, p_1, p_2)_{\geq 0}$, and then we show that these polynomials are constructible. Obviously, there exists $\varepsilon_0 > 0$ such that

$$\forall\, F \in \mathcal{F}_{d-1}(P)\ \forall\, G \in \mathcal{F}(P)\ :\ F \cap G = \emptyset \implies \operatorname{aff} F \cap (G + \mathbb{B}^d(o, \varepsilon_0)) = \emptyset. \tag{7.2}$$

In view of Theorem 6.1(II), condition (6.2) is fulfilled for all sufficiently small $\rho > 0$. We also assume that ρ is sufficiently small so that (5.3) in Lemma 5.5 is fulfilled. For

$v \in \operatorname{vert}(P)$ and $l \in \mathbb{N}$ let the polynomials $b_{v,l}(x)$ be as in Proposition 6.3. Fix $\alpha_l > 0$ satisfying

$$|b_{w,l}(x)| \le \alpha_l \qquad \forall\, x \in P + \mathbb{B}^3(o, \varepsilon_0)\ \forall\, w \in \operatorname{vert}(P). \tag{7.3}$$

We define

$$\begin{aligned} p_0(x) &:= g_m(x), \\ p_1(x) &:= \sum_{v \in \operatorname{vert}(P)} f_v(x)^{2k} b_{v,l}(x), &(7.4) \\ p_2(x) &:= \prod_{F \in \mathcal{F}_2(P)} q_F(P, x), &(7.5) \end{aligned}$$

where

$$f_v(x) := \prod_{F \in \mathcal{F}_2(P) \setminus \mathcal{F}_2(v,P)} q_F(P, x) \tag{7.6}$$

for every $v \in \operatorname{vert}(P)$ and the parameters $k, m \in \mathbb{N}$ will be fixed later. It will be shown that for a sufficiently large k and m we have $P = (p_0, p_1, p_2)_{\ge 0}$. Let us first show that there exists $\varepsilon \in (0, \varepsilon_0]$ such for all sufficiently large $k \in \mathbb{N}$ we have

$$U(P, \varepsilon, \rho) \cap (p_1, p_2)_{\ge 0} \subseteq P \cup \left(\bigcup_{v \in \operatorname{vert}(P)} \big(v - \operatorname{int} S_\rho(P, v)\big) \right). \tag{7.7}$$

Let us consider an arbitrary $x \in U(P, \varepsilon_0, \rho)$.

Case 1: $x \in U_P(P, \varepsilon_0, \rho) = P$. Clearly, x belongs to the left and the right hand side of (7.7).

Case 2: $x \in U_F(P, \varepsilon_0, \rho)$ *for some* $F \in \mathcal{F}_2(P)$. We have $p_2(x) \le 0$ with equality if and only if $x \in F$. Consequently $U_F(P, \varepsilon_0, \rho) \cap (p_2)_{\ge 0} \subseteq P$.

Case 3: $x \in U_I(P, \varepsilon_0, \rho)$ *for some* $I \in \mathcal{F}_1(P)$. If $x \in I$ and $v \in \operatorname{vert}(P) \setminus \operatorname{vert}(I)$, then at least one of the two facets F from $\mathcal{F}_2(I, P)$ satisfies $v \notin F$, which yields $f_v(x) = 0$. Consequently

$$\max_{v \in \operatorname{vert}(P) \setminus \operatorname{vert}(I)} |f_v(x)| = 0 \qquad \forall\, x \in I \tag{7.8}$$

We show that there exists $\beta_I > 0$ is such that

$$\beta_I \le \max_{w \in \operatorname{vert}(I)} |f_w(x)| \qquad \forall\, x \in U_I(P, \varepsilon_0, \rho). \tag{7.9}$$

Choose $x \in U_I(P, \varepsilon_0, \rho)$. If $x \in \operatorname{vert}(I)$, then $f_w(x) > 0$ for $w = x$. If $x \in \operatorname{relint} I$, then $f_w(x) > 0$ for every $w \in \operatorname{vert}(I)$. Now assume $x \in U_I(P, \varepsilon_0, \rho) \setminus I$. We fix arbitrary $w \in \operatorname{vert}(I)$ and $F \in \mathcal{F}_2(P) \setminus \mathcal{F}_2(w, P)$. Consider the subcase $F \cap I = \emptyset$. By the definition of $U_I(P, \varepsilon_0, \rho)$, we have $x \in I + \mathbb{B}^3(o, \varepsilon_0)$, and by (7.2), we obtain $q_F(P, x) \neq 0$. Consider the subcase $F \cap I \neq \emptyset$. We denote by v the endpoint of I distinct from w. Then $F \in \mathcal{F}_2(v, P)$. Let us show that $q_F(P, x) \neq 0$ by contradiction. Assume the contrary, i.e., $q_F(P, x) = 0$. Then $F \in \mathcal{F}^-(P, x)$. Furthermore, $\mathcal{F}_2(I, P) \subseteq \mathcal{F}^-(P, x)$, by the definition of $U_I(P, \varepsilon_0, \rho)$. Taking into account the above relations together with $\mathcal{F}_2(I, P) \subseteq \mathcal{F}_2(v, P)$ and $F \in \mathcal{F}_2(v, P)$ we deduce $\mathcal{F}_2(I, P) \cup \{F\} \subseteq \mathcal{F}^-(P, x) \cap \mathcal{F}_2(v, P)$. Hence $|\mathcal{F}^-(P, x) \cap \mathcal{F}_2(v, P)| \ge 3$, which implies that (5.3) cannot be fulfilled, a contradiction. Summarizing we get

$$\max_{w \in \operatorname{vert}(I)} |f_w(x)| > 0 \qquad \forall\, x \in U_I(P, \varepsilon_0, \rho),$$

which yields the existence of β_I satisfying (7.9). By Proposition 6.3(III)

$$b_{w,l}(x) = 0 \qquad \forall\, w \in \operatorname{vert}(I)\ \forall\, x \in I. \tag{7.10}$$

If $x \in \operatorname{aff} I \setminus I$, one has $x \notin S_\rho(P, w)$ for some vertex of $w \in \operatorname{vert}(I)$. Consequently $U_I(P, \varepsilon_0, \rho) \cap \operatorname{aff} I = I$. By Proposition 6.3(III) we get

$$b_{w,l}(x) < 0 \qquad \forall\, w \in \operatorname{vert}(I)\ \forall\, x \in U_I(P, \varepsilon_0, \rho) \setminus I, \tag{7.11}$$

We also notice that $\max_{v \in \operatorname{vert}(P) \setminus \operatorname{vert}(I)} |f_v(x)|$ and $b_{w,l}(x)$ are semi-algebraic functions and $U_I(P, \varepsilon_0, \rho)$ is a semi-algebraic set. Hence, taking into account (7.8), (7.10) and (7.11) and applying Theorem 3.1, we obtain the existence of $k_I \in \mathbb{N}$ and $\gamma_I > 0$ such that

$$\max_{v \in \operatorname{vert}(P) \setminus \operatorname{vert}(I)} f_v(x)^{2k_I} \le \gamma_I \min_{w \in \operatorname{vert}(I)} |b_{w,l}(x)| \tag{7.12}$$

for every $x \in U_I(P, \varepsilon_0, \rho)$. In view of (7.8), we can choose $\varepsilon_I \in (0, \varepsilon_0]$ such that for every $x \in U_I(\varepsilon_I, \rho)$ one has

$$\max_{v \in \operatorname{vert}(P) \setminus \operatorname{vert}(I)} |f_v(x)| \le \frac{\beta_I}{2}. \tag{7.13}$$

We also assume that k is large enough so that the inequality

$$\alpha_l\, |\operatorname{vert}(P)|\, \gamma_I\, 2^{-2k} \left(\frac{2}{\beta_I}\right)^{2k_I} \le \frac{1}{2} \tag{7.14}$$

is fulfilled.

Then for every $x \in U_I(P, \rho, \varepsilon_I)$ and every $k \in \mathbb{N}$ with $k \ge k_I$ we obtain

$$\begin{aligned}
&\left| \sum_{v \in \operatorname{vert}(P) \setminus \operatorname{vert}(I)} f_v(x)^{2k} b_{v,l}(x) \right| \\
\overset{(7.3)}{\le}\ & \alpha_l \sum_{v \in \operatorname{vert}(P) \setminus \operatorname{vert}(I)} f_v(x)^{2k} \\
\le\ & \alpha_l\, |\operatorname{vert}(P)| \max_{v \in \operatorname{vert}(P) \setminus \operatorname{vert}(I)} f_v(x)^{2k} \\
\overset{(7.12)}{\le}\ & \alpha_l\, |\operatorname{vert}(P)|\, \gamma_I \min_{w \in \operatorname{vert}(I)} |b_{w,l}(x)| \max_{v \in \operatorname{vert}(P) \setminus \operatorname{vert}(I)} f_v(x)^{2(k-k_I)} \\
\overset{(7.13)}{\le}\ & \alpha_l\, |\operatorname{vert}(P)|\, \gamma_I \left(\frac{\beta_I}{2}\right)^{2(k-k_I)} \min_{w \in \operatorname{vert}(I)} |b_{w,l}(x)| \\
\overset{(7.9)}{\le}\ & \alpha_l\, |\operatorname{vert}(P)|\, \gamma_I\, 2^{-2k} \left(\frac{2}{\beta_I}\right)^{2k_I} \min_{w \in \operatorname{vert}(I)} |b_{w,l}(x)| \max_{w \in \operatorname{vert}(I)} f_w(x)^{2k} \\
\overset{(7.14)}{\le}\ & \frac{1}{2} \min_{w \in \operatorname{vert}(I)} |b_{w,l}(x)| \max_{w \in \operatorname{vert}(I)} f_w(x)^{2k} \\
\le\ & \frac{1}{2} \min_{w \in \operatorname{vert}(I)} |b_{w,l}(x)| \sum_{w \in \operatorname{vert}(I)} f_w(x)^{2k} \\
\le\ & \frac{1}{2} \sum_{w \in \operatorname{vert}(I)} f_w(x)^{2k}\, |b_{w,l}(x)|.
\end{aligned}$$

Consequently, for x and k as above, we get

$$\begin{aligned} p_1(x) &\leq \sum_{w\in\mathrm{vert}(I)} f_w(x)^{2k} b_{w,l}(x) + \left| \sum_{v\in\mathrm{vert}(P)\setminus\mathrm{vert}(I)} f_v(x)^{2k} b_{v,l}(x) \right| \\ &\leq \sum_{w\in\mathrm{vert}(I)} f_w(x)^{2k} b_{w,l}(x) + \frac{1}{2} \sum_{w\in\mathrm{vert}(I)} f_w(x)^{2k} \left| b_{w,l}(x) \right| \\ &\overset{(7.11)}{=} \frac{1}{2} \sum_{w\in\mathrm{vert}(I)} f_w(x)^{2k}\, b_{w,l}(x) \overset{(7.11)}{\leq} 0 \end{aligned}$$

with equality $p_1(x) = 0$ if and only if $x \in I$.

Case 4: $x \in U'_v(P, \varepsilon_0, \rho)$ for some $v \in \mathrm{vert}(P)$. By Proposition 6.3(V) and the definition of $U'_v(P, \varepsilon_0, \rho)$ we have

$$b_{v,l}(x) < 0 \qquad \forall\, x \in U'_v(P, \varepsilon_0, \rho) \setminus \{v\}. \tag{7.15}$$

In view of (7.2), the definition of $f_v(x)$, and the inclusion $U'_v(P, \varepsilon_0, \rho) \subseteq \mathbb{B}^3(v, \varepsilon_0)$, there exists $\beta_v > 0$ such that

$$f_v(x) \geq \beta_v \qquad \forall\, x \in U'_v(P, \varepsilon_0, \rho). \tag{7.16}$$

On the other hand

$$\max_{w\in\mathrm{vert}(P)\setminus\{v\}} |f_w(v)| = 0. \tag{7.17}$$

Notice that $\max_{w\in\mathrm{vert}(P)\setminus\{v\}} |f_w(x)|$ and $b_{v,l}(x)$ are semi-algebraic function, and $U'_v(P, \varepsilon_0, \rho)$ is a semi-algebraic set. Thus, taking into account (7.15) and (7.17) and applying Theorem 3.1 we find $\gamma_v > 0$ and $k_v \in \mathbb{N}$ such that

$$\max_{w\in\mathrm{vert}(P)\setminus\{v\}} |f_w(x)|^{2k_v} \leq \gamma_v\, |b_{v,l}(x)| \tag{7.18}$$

for every $x \in U'_v(P, \varepsilon_0, \rho)$. In view of (7.17), we can choose $\varepsilon_v \in (0, \varepsilon_0]$ such that

$$\max_{w\in\mathrm{vert}(P)\setminus\{v\}} |f_w(x)| \leq \frac{\beta_v}{2} \tag{7.19}$$

for every $x \in U'_v(P, \varepsilon_v, \rho)$. We assume that k is large enough so that

$$\alpha_l\, |\,\mathrm{vert}(P)|\, \gamma_v\, \left(\frac{\beta_v}{2}\right)^{-2k_v} 2^{-2k} \leq \frac{1}{2}. \tag{7.20}$$

Then for every $x \in U'_v(P, \varepsilon_v, \rho)$ and $k \geq k_v$ as above we obtain

$$\begin{aligned}
\left| \sum_{w \in \mathrm{vert}(P) \setminus \{v\}} f_w(x)^{2k} b_{w,l}(x) \right| &\overset{(7.3)}{\leq} \alpha_l \, |\mathrm{vert}(P)| \max_{w \in \mathrm{vert}(P)\setminus\{v\}} |f_w(x)|^{2k} \\
&\overset{(7.18)}{\leq} \alpha_l \, |\mathrm{vert}(P)| \, \gamma_v \, |b_{v,l}(x)| \max_{w \in \mathrm{vert}(P)\setminus\{v\}} |f_w(x)|^{2(k-k_v)} \\
&\overset{(7.19)}{\leq} \alpha_l \, |\mathrm{vert}(P)| \, \gamma_v \left(\frac{\beta_v}{2}\right)^{2(k-k_v)} |b_{v,l}(x)| \\
&\overset{(7.16)}{\leq} \alpha_l \, |\mathrm{vert}(P)| \, \gamma_v \left(\frac{\beta_v}{2}\right)^{2(k-k_v)} (\beta_v)^{-2k} f_v(x)^{2k} |b_{v,l}(x)| \\
&= \alpha_l \, |\mathrm{vert}(P)| \, \gamma_v \left(\frac{\beta_v}{2}\right)^{-2k_v} 2^{-2k} f_v(x)^{2k} |b_{v,l}(x)| \\
&\overset{(7.20)}{\leq} \frac{1}{2} f_v(x)^{2k} |b_{v,l}(x)|.
\end{aligned}$$

Hence, taking into account (7.15) and (7.16), we obtain that for k satisfying (7.20) and all $x \in U'_v(P, \varepsilon_v, \rho)$ we have $p_1(x) \leq 0$ with equality if and only if $x = v$. Consequently, for x and $k \geq k_v$ as above we obtain

$$\begin{aligned}
p_1(x) &\leq f_v(x)^{2k} b_{v,l}(x) + \left| \sum_{w \in \mathrm{vert}(P)\setminus\{v\}} f_w(x)^{2k} b_{w,l}(x) \right| \leq f_v(x)^{2k} b_{v,l}(x) + \frac{1}{2} f_v(x)^{2k} |b_{v,l}(x)| \\
&\overset{(7.15)}{=} \frac{1}{2} f_v(x)^{2k} b_{v,l}(x) \overset{(7.15)}{\leq} 0
\end{aligned}$$

with equality $p_1(x) = 0$ if and ony if $x = v$.

Case 5: $x \in U_v(P, \varepsilon_0, \rho) \setminus U'_v(P, \varepsilon_0, \rho)$ for some $v \in \mathrm{vert}(P)$. By the definition of $U_v(P, \varepsilon_0, \rho)$ and $U'_v(P, \varepsilon_0, \rho)$ we easily see that $x \in v - \mathrm{int}\, S_\rho(P, v)$. Thus, $U_v(P, \varepsilon_0, \rho) \setminus U'_v(P, \varepsilon_0, \rho) \subseteq v - \mathrm{int}\, S_\rho(P, v)$.

By means of the arguments given in the above five cases we verified that (7.7) holds if $k \geq k_v$, $k \geq k_I$, $\varepsilon \leq \varepsilon_v$, $\varepsilon \leq \varepsilon_I$ for all $v \in \mathrm{vert}(P)$ and $I \in \mathcal{F}_1(P)$ and the inequalities (7.14) and (7.20) are fulfilled. By Proposition 5.1, there exists $\delta > 0$ such that $P + \mathbb{B}^3(o, \delta) \subseteq U(P, \varepsilon, \rho)$. By condition $\mathcal{A}(P)$, we can choose $m \in \mathbb{N}$ such that $(p_0)_{\geq 0} \subseteq P + \mathbb{B}^3(o, \delta)$. Thus, for m as above we obtain

$$\begin{aligned}
(p_0, p_1, p_2)_{\geq 0} &= (p_0, p_1, p_2)_{\geq 0} \cap (P + \mathbb{B}^3(o, \delta)) \subseteq (p_0, p_1, p_2)_{\geq 0} \cap U(P, \varepsilon, \rho) \\
&\overset{(7.7)}{\subseteq} (p_0)_{\geq 0} \cap \left(P \cup \bigcup_{v \in \mathrm{vert}(P)} (v - \mathrm{int}\, S_\rho(P, v)) \right) \\
&= ((p_0)_{\geq 0} \cap P) \cup \left((p_0)_{\geq 0} \cap \bigcup_{v \in \mathrm{vert}(P)} (v - \mathrm{int}\, S_\rho(P, v)) \right) \\
&\overset{(6.2)}{=} P \cup \mathrm{vert}(P) = P.
\end{aligned}$$

Thus $(p_0, p_1, p_2)_{\geq 0} = P$.

Now let us show the existence of an algorithm constructing $p_1(x)$, $p_2(x)$, $p_3(x)$ as above. The constructibility of $q_2(x)$ is obvious. Below we show how appropriate polynomials $p_0(x)$ and $p_1(x)$ can be determined. The formulas for $p_0(x)$ and $p_1(x)$ involve

the parameters $l, m, k \in \mathbb{N}$. It is not difficult to construct the sequences $(l_j)_{j=1}^{\infty}$, $(m_j)_{j=1}^{\infty}$, $(k_j)_{j=1}^{\infty}$ such that

$$\mathbb{N}^3 = \{(l_j, m_j, k_j) : j \in \mathbb{N}\} .$$

We proceed as follows.

1: Set $j := 1$.

2: Set $l := l_j$, $m := m_j$, $k := k_j$.

3: Determine $p_0(x)$ and $p_1(x)$ by (7.1) and (7.4), respectively.

4: If $P \neq (p_0, p_1, p_2)_{\geq 0}$, set $j := j + 1$ and go to Step 2.

5: Return $p_0(x)$ and $p_1(x)$.

We remark that at Step 4 the comparison of P and $(p_0, p_1, p_2)_{\geq 0}$ can be performed algorithmically, which follows directly from Theorem 3.2. □

7.2 The case of unbounded polyhedra

Proof of Theorem 1.1. The case when P is bounded follows directly from Theorem 7.2. Let us consider the case when P is unbounded. Every polyhedron P can be represented as a sum $P = Q+L$, where L is an affine space and Q is a line-free polyhedron such that $\operatorname{aff} Q$ is orthogonal to L; see (4.2). If one can construct polynomials $p_0(x), \ldots, p_{d-1}(x) \in \mathbb{R}[x]$ with $Q = \{x \in \operatorname{aff} Q : p_0(x) \geq 0, \ldots, p_{d-1}(x) \geq 0\}$, then

$$P = \left\{x \in \mathbb{R}^d : p_0(x \mid \operatorname{aff} Q) \geq 0, \ldots, p_{d-1}(x \mid \operatorname{aff} Q)\right\} ,$$

where $x|\operatorname{aff} Q$ is the orthogonal projection of x onto $\operatorname{aff} Q$. Thus, also P can be represented by d polynomial inequalities. Consequently, we can restrict ourselves to the case when P is a d-dimensional line-free polyhedron in $\mathbb{R}^d$.

From now on, we replace P by an isometric copy of P in $\mathbb{R}^{d+1}$ and also assume that $o \notin \operatorname{aff} P$. Then $\hom(P)$ is a $(d+1)$-dimensional pointed polyhedral cone. Since $\hom(P)$ is pointed, one can determine a hyperplane H' in $\mathbb{R}^{n+1}$ with $o \notin H'$ such that $P' := \hom(P) \cap H'$ is bounded and $\hom(P) = \operatorname{cone}(P')$. We apply Theorem 6.1(III) to the polytope P' and construct a sequence of polynomials $(g_l(x))_{l=1}^{\infty}$ satisfying conditions $\mathcal{A}(P')$, $\mathcal{I}(P')$, $\mathcal{C}(P')$, and $\mathcal{H}(P')$. By Theorem 7.2 one can construct polynomials $f_1(x), \ldots, f_{d-1}(x)$ satisfying

$$P' = \{x \in H' : f_0(x) \geq 0, \ldots, f_{d-1}(x) \geq 0\}$$

and $f_0(x) = g_l(x)$ for some $l \in \mathbb{N}$. We choose an affine function $f(x)$ on H' such that $\{x \in H' : f(x) = 0\}$ is a $(d-1)$-dimensional affine subspace of H' and such that $f(x)$ is strictly positive on the set $\{x \in H' : p_0(x) \geq 0\}$; the above choice is possible since $\{x \in H' : p_0(x) \geq 0\}$ is bounded. For $j = 1, \ldots, d-1$ let us define $k_j := 1$ if $f_j(x)$ has odd degree and $k_j := 0$, otherwise. We set $p_j(x) := f_j(x)\, f(x)^{k_j}$ for $j = 1, \ldots, d-1$ and $p_0(x) := f_0(x)$. By construction, the polynomials $p_0(x), \ldots, p_{d-1}(x)$ have even degrees. We have $P' = \{x \in H' : p_0(x) \geq 0, \ldots, p_{d-1}(x) \geq 0\}$. In fact, if $f(x) > 0$, for every $j = 0, \ldots, d-1$ the inequality $f_j(x) \geq 0$ is equivalent to the inequality $p_j(x) \geq 0$ and otherwise $p_0(x) < 0$, by the choice of $f(x)$ and the definition of $p_0(x)$. For $j = 0, \ldots, d-1$ let $\tilde{p}_j(x)$ be

the homogenization of $p_j(x)|_{H'}$. Let $u' \in \mathbb{R}^d \setminus \{o\}$ denote a normal of H'. By the definition of the homogeneous continuation and the evenness of the degrees of $p_0(x), \ldots, p_{d-1}(x)$, we have

$$\left\{x \in \mathbb{R}^{d+1} : \tilde{p}_0(x) \geq 0, \ldots, \tilde{p}_{d-1}(x) \geq 0,\ \langle x, u' \rangle \neq 0\right\} \cup \{o\} = \hom(P) \cup (-\hom(P)).$$

Furthermore, since $g_l(x)$ satisfies $\mathcal{H}(P)$, if $\langle x, u' \rangle = 0$, then $\tilde{p}_0(x) \leq 0$ with equality if and only if on $x = o$. Hence

$$\left\{x \in \mathbb{R}^{d+1} : \tilde{p}_0(x) \geq 0, \ldots, \tilde{p}_{d-1}(x) \geq 0\right\} = \hom(P) \cup (-\hom(P)),$$

which implies

$$\begin{aligned}&\{x \in \operatorname{aff} P : \tilde{p}_0(x) \geq 0, \ldots, \tilde{p}_{d-1}(x) \geq 0\} = \big(\hom(P) \cup (-\hom(P))\big) \cap (\operatorname{aff} P)\\ =&\big(\operatorname{cone}(P) \cup \operatorname{rec}(P) \cup (-\operatorname{cone}(P)) \cup (-\operatorname{rec}(P))\big) \cap (\operatorname{aff} P) = P.\end{aligned}$$

and we are done. □

We wish to give another proof of Theorem 1.1. First we formulate a modified version of Łojasiewicz's Inequality.

Lemma 7.3. *Let A be a bounded and closed semi-algebraic set in $\mathbb{R}^d$. Let $f(x)$ and $g(x)$ be continuous, semi-algebraic functions on A satisfying*

$$\{x \in A : f(x) = 0\} \subseteq \{x \in A : g(x) = 0\}.$$

Then there exists $n' \in \mathbb{N}$ such that for every $n \in \mathbb{N}$ with $n \geq n'$ there is a constant $\lambda > 0$ satisfying

$$|g(x)|^n \leq \lambda\, |f(x)|$$

for every $x \in A$.

Proof. By Theorem 3.1 we can fix $n' \in \mathbb{N}$ and $\lambda' > 0$ such that $|g(x)|^{n'} \leq \lambda' |f(x)|$ for every $x \in A$. Let $\mu \geq 0$ be an upper bound of $|g(x)|$ on A. Then for every $n \in \mathbb{N}$ with $n \geq n'$ we have $|g(x)|^n \leq \mu^{n-n'} \lambda' |f(x)|$, so that we may set $\lambda := \mu^{n-n'} \lambda'$. □

The following proposition (which is also interesting on itself) can be used to give another proof of Theorem 1.1.

Proposition 7.4. *Let $d \in \mathbb{N}$ and $d \geq 2$. Assume that there exists an algorithm that takes an arbitrary n-polytope P in $\mathbb{R}^d$ and constructs polynomials $p_0(x), \ldots, p_{n-1}(x) \in \mathbb{R}[x]$ satisfying $P = \{x \in \operatorname{aff} P : p_0(x) \geq 0, \ldots, p_{n-1}(x) \geq 0\}$ and $p_0(x) > 0$ for all $x \in P \setminus$* $\operatorname{vert}(P)$. *Then there exists an algorithm that takes an arbitrary d-polyhedron P in $\mathbb{R}^d$ and constructs $f_0(x), \ldots, f_{d-1}(x) \in \mathbb{R}[x]$ satisfying $P = (f_0, \ldots, f_{d-1})_{\geq 0}$.*

Proof. Consider an arbitrary unbounded d-polyhedron P in $\mathbb{R}^d$. Using the same arguments as in the beginning of the proof of Theorem 1.1 we may restrict ourselves to the case when P is line-free.

The set $\hom(P)$ is a $(d+1)$-dimensionial pointed polyhedral cone. Since $\hom(P)$ is pointed, one can determine a hyperplane H' in $\mathbb{R}^{n+1}$ such that $P' := \hom(P) \cap H'$ is bounded and $\hom(P) = \operatorname{cone}(P')$. Let $u' \in \mathbb{R}^{d+1} \setminus \{o\}$ be the normal of H' with

$H' = \left\{x \in \mathbb{R}^{d+1} : \langle x, u' \rangle = 1\right\}$. Let us consider polynomials $p_0(x), \ldots, p_{d-1}(x) \in \mathbb{R}[x]$ such that

$$P = \{x \in \operatorname{aff} P' : p_0(x) \geq 0, \ldots, p_{d-1}(x) \geq 0\}$$

and $p_0(x) > 0$ for every $x \in P \setminus \operatorname{vert}(P)$. Without loss of generality we may assume that $p_0(x), \ldots, p_{d-1}(x)$ are homogeneous. We define

$$f(x) := \prod_{v \in \operatorname{vert}(P')} \left(\|x\|^2 \|v\|^2 - \langle x, v \rangle^2\right).$$

By Lemma 7.3 applied to $f(x)$ and $p_0(x)$ restricted to P', there exist $\lambda > 0$ and $l \in \mathbb{N}$ satisfying $l \deg f(x) > \deg p_0(x)$ and $\lambda\, p_0(x) - f(x)^l \geq 0$ for every $x \in P'$ (where deg stands for degree).

Define $f_0(x) := \lambda\, p_0(x)\, \langle x, u' \rangle^{k_0} - f(x)^l$ where $k_0 \in \mathbb{N}$ is chosen in such a way that f_0 is homogeneous, i.e., k_0 is determined from the equality

$$k_0 + \deg p_0(x) = l \deg f(x). \tag{7.21}$$

We also set $f_i(x) := p_i(x)\, \langle x, u' \rangle^{k_i}$ for $i = 1, \ldots, d-1$, where $k_1, \ldots, k_{d-1} \in \{1, 2\}$ are chosen in such a way that $f_1(x), \ldots, f_{d-1}(x)$ have even degrees. By construction, the polynomials $f_0(x), \ldots, f_{d-1}(x)$ are homogeneous, have even degrees, and satisfy

$$\hom(P) \cup (-\hom(P)) = \left\{x \in \mathbb{R}^{d+1} : f_0(x) \geq 0, \ldots, f_{d-1}(x) \geq 0\right\}$$

Hence

$$\{x \in \operatorname{aff} P : f_0(x) \geq 0, \ldots,\ f_{d-1}(x) \geq 0\} = \big(\hom(P) \cup (-\hom(P))\big) \cap (\operatorname{aff} P) = P.$$

It remains to show that $\lambda > 0$ and s are constructible. One can construct sequences $(\lambda_j)_{j=1}^{\infty}$ and $(l_j)_{j=1}^{\infty}$ such that

$$\{(\lambda, l) : \lambda \in \mathbb{N},\ l \in \mathbb{N},\ l \deg f(x) > \deg p_0(x)\} = \{(\lambda_j, l_j) : j \in \mathbb{N}\}.$$

Thus, we can proceed as follows

1: Define $k_1, \ldots, k_{d-1}$ as explained above.

2: Set $j := 1$.

3: Set $\lambda := \lambda_j$ and $l := l_j$.

4: Determine k_0 from (7.21).

5: Set $f_0(x) := \lambda\, p_0(x)\, \langle x, u' \rangle^{k_0} - f(x)^l$.

6: If $f_0(x) < 0$ for some $x \in P$, set $j := j + 1$ and go to Step 3.

7: Return $f_0(x)$ and stop.

In view Theorem 3.2, Step 6 can be implemented algorithmically. □

Theorem 1.1 is a direct consequence of Theorem 7.2 and Proposition 7.4. The advantage of the proof of Theorem 1.1 with the help of Proposition 7.4 is that it does not use much of the structure of the "input" polynomials. The disadvantage is that the proof of Proposition 7.4 is less direct, since it uses Łojasiewicz's Inequality and involves an "exhaustive" search of appropriate parameters.

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Gennadiy Averkov and Martin Henk,
Universitätsplatz 2, Institut für Algebra und Geometrie,
Fakultät für Mathematik, Otto-von-Guericke-Universität Magdeburg,
D-39106 Magdeburg
e-mails: gennadiy.averkov@googlemail.com
martin.henk@mathematik.uni-magdeburg.de
web: http://fma2.math.uni-magdeburg.de/∼averkov
http://fma2.math.uni-magdeburg.de/∼henk